Chapter 1

On the origin of Darcy's law[1]

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When one thinks about porous media, the immediate concepts that come to mind are porosity, permeability and Darcy's law. The latter is probably one of the most important tools in hydrodynamics and engineering presenting numerous applications in oil recovery, chemical engineering, nuclear safety and many other engineering field. It has been first formulated by the French engineer Henri Darcy in 1856 in Appendix D of his famous book *Fontaines publiques de la ville de Dijon* [2] to describe flow through beds of sand. His column experiments (see Figure 1.1) showed that the total discharge is proportional to the total pressure drop divided by the fluid viscosity and that the linear coefficient, called permeability, is intrinsic to the investigated porous medium. Darcy's law is the equation that governs the flow of fluid in porous media. If one opens any fluid dynamics textbook, however, one learns that fluid motion is described by the Navier-Stokes equations. So, what is this law? Are there any links between Darcy's law and fluid mechanics theory? This Chapter introduces two different representations of the physics of fluid flow in porous media and their interrelation. These different approaches will lead us to the very definition of a porous medium.

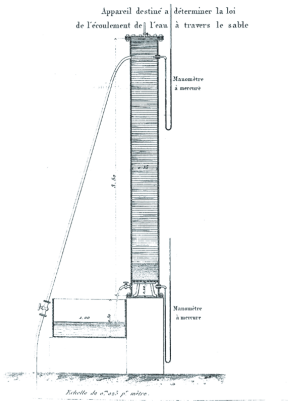

Figure 1.1: Henri Darcy and its famous sand column experiment

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1.1 Two representations of the physics of fluid flow in porous media

A porous medium is a material that contains void spaces occupied by one or more fluid phases (gas, water, oil, etc.) and a solid matrix. A two-dimensional (2D) cross-section through the three-dimensional (3D) image of a sandstone is presented in Figure 1.2. The solid grains appear in white and the void or pore spaces are black. A fluid, denoted β, occupies pore space and we define this region as V_β as opposed to V_σ, the region occupied by the solid phase, σ. Figure 1.2 clearly highlights a sharp delineation between the solid and the void space: the interfacial area, denoted $A_{\beta\sigma}$. A lot of physico-chemical phenomena, such as surface reaction, may occur at this interface.

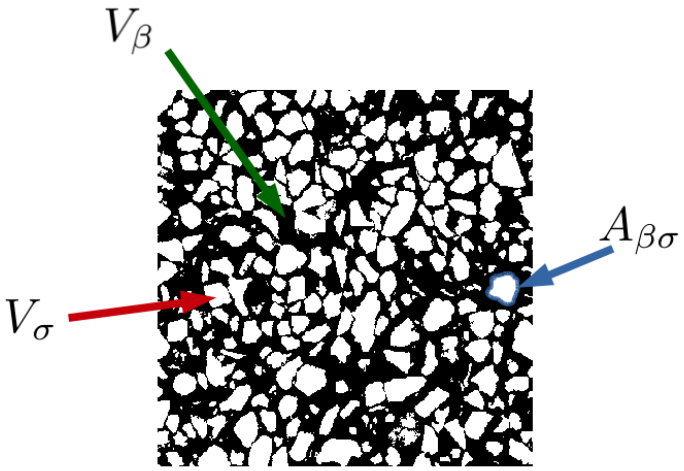

Figure 1.2: Two-dimensional slice of a sandstone. The black region represents the void space while the white region represents the solid grains.

Figure 1.2 also displays the high complexity of the pore network topology and morphology of real rocks. The local heterogeneities of the connected pore structure strongly influence the flow pathways. In order to characterize the flow in every location in V_β, we can define a velocity, $\mathbf{v}_\beta$, and a pressure, p_β. Figure 1.3a, shows that a fluid flows through some preferential pathways leaving some area of the pore space stagnant. In natural porous media such as underground reservoirs, the flow is usually very slow. In that case, and assuming incompressible and Newtonian fluids, the flow patterns are independent of the flow rate and the type of the fluid. Only the magnitude of $\mathbf{v}_\beta$ changes linearly with the pressure drop. Actually, there are universal conservation laws that govern the fluid displacement in such conditions which are called the Stokes equations, a subset of the more general equations of fluid mechanics, the Navier-Stokes equations. For a given fluid of density ρ_β and viscosity μ_β, these equations read,

$$\begin{cases} \nabla.\mathbf{v}_\beta = 0 & \text{in } V_\beta \\ 0 = -\nabla p_\beta + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \mathbf{v}_\beta & \text{in } V_\beta \end{cases} \tag{1.1}$$

For the time being, you can take these equations for granted. Further details and explanations are given in the next chapter. To be well-posed, this mathematical problem has to be completed by defining boundary conditions at the solid walls, $A_{\beta\sigma}$. For fluid flow, we consider no-slip conditions at the fluid/solid interface:

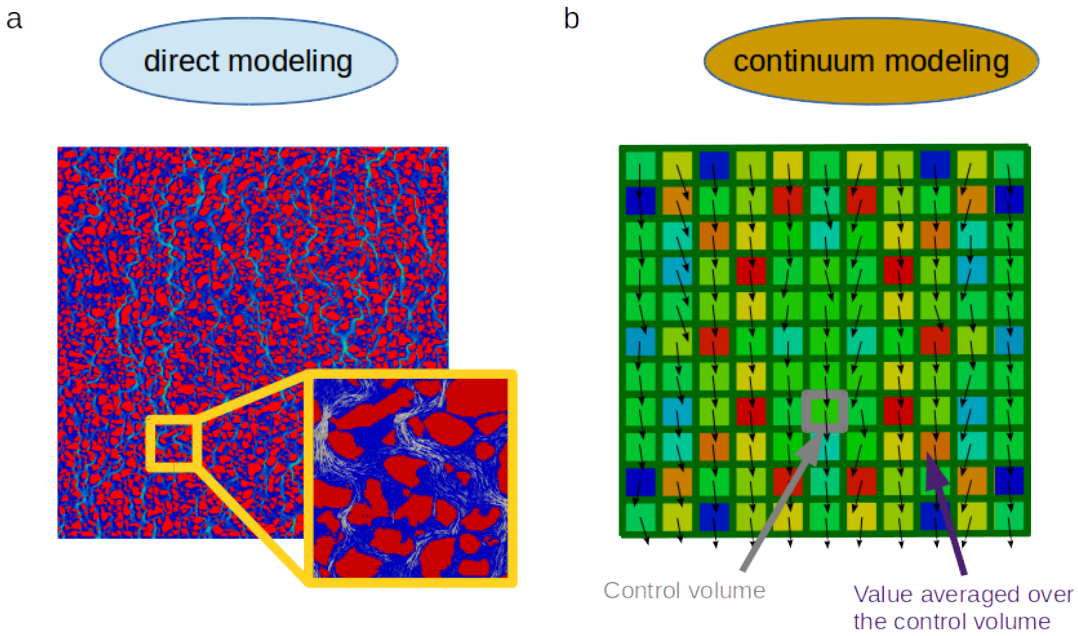

Figure 1.3: Two different representations of the physics of flow in porous media. The arrows represent the velocity vectors. (a) direct modeling or pore-scale approach where the solid is explicitly represented. (b) the continuum modeling or Darcy scale where the physics is governed by quantities averaged over control volumes. The color map represents the volume fraction of solid per control volume.

for every point of $A_{\beta\sigma}$, $\mathbf{v}_\beta = 0$.

Solving the flow in a porous medium with Stokes equations is called *direct modeling* approach because the physics is directly solved without any assumptions. The only input is the geometry of the solid skeleton and the fluid properties (viscosity and density). It is worth mentioning that for complex topologies Stokes equations do not have simple analytical solutions and need therefore to be approximated numerically. This approach is often referred to as *direct numerical simulation* (*DNS*). Despite its accuracy and close representation of the physics, this approach has very severe limitation. Indeed, the geometry of the complex pore space enclosed by the solid skeleton must be described in complete detail. Although the last decade has seen tremendous progress in high resolution imaging techniques such as X-ray computed microtomography, this approach is not sustainable for large domains. It is of particular importance when investigating natural porous media. Even if you had the most powerful computer, how could you get the exact pore network topology of $100\,\mathrm{m}^3$ of rocks? As this representation of the physics of fluid flow in porous material often applies to small objects whose exact pore structure can be resolved, it is often referred to as *pore-scale modeling* or *pore-scale simulation*.

Another representation of the physics of fluid flow in a porous medium allows to deal with larger domains of investigation.. In this representation, control volumes are defined all over the domain and quantities averaged over these volumes are considered (see Figure 1.3b). We will see in the next chapter that the control volume can not be taken arbitrarily since it needs to be a Representative Elementary Volume (REV). The very definition of the REV is often an integral part of the modeling challenge. Nevertheless, assuming that the REV scale is known, we can define quantities averaged according to this control volume. With this

approach, the pore space is no longer explicitely represensted and we have instead information such as the volume fraction of void in a control volume. From now on, the fluid flow unknowns are the average velocity $\langle \mathbf{v}_\beta \rangle$ and the average pressure $\langle p_\beta \rangle^\beta$. These bracket notations are used to emphasize the differences between this representation and the direct modeling approach. We will see in the Section on volume averaging how to compute these quantities. Because the pore network topology of the material is not explicit the fluid mechanics equations are no longer valid. In this representation, the averaged quantities are governed by Darcy's law,

$$\begin{cases} \nabla . \langle \mathbf{v}_\beta \rangle = 0, \\ \langle \mathbf{v}_\beta \rangle = -\frac{\mathsf{K}}{\mu_\beta} . \left(\nabla \langle p_\beta \rangle^\beta - \rho_\beta \mathbf{g} \right). \end{cases} \tag{1.2}$$

where the tensor K is the permeability, a constant intrinsic to the porous medium under investigation. It contains the information related to the pore space topology, or at least enough information to be representative of its geometry. Permeability is referred to as an *effective coefficient* because it translates information from a smaller scale. We call this approach *continuum* or *Darcy modeling*. We also name it *macroscale* modeling as opposed to *microscale* modeling. Without going any further, we see that if one can get the pore space geometry of a REV and solves the flow in this REV, then by averaging the resulting velocity profile one can deduce the permeability.

Ideally, these two representations of the physics should produce the same results. They present, however, some fundamental differences in their formalism. The direct modeling has an explicit interface that delineates the solid and the void space where boundary conditions need to be specified and each point of the direct model domain contains either fluid or solid. Conversely, each point of the domain in the continuum model contains both fluid and solid in the average sense. With such an approach, all the solid/fluid interfacial information is embedded in effective properties.

1.2 Volume averaging: from Stokes to Darcy

We have seen in the previous section that the physics of flow in porous media can be represented through two different approaches: direct and continuum modeling. The latter being an average version of the former. In this section, we provide a mathematical definition for the average quantities and we explain the origin of Darcy's law from the pore-scale physics.

Let's define a function, ϕ_β, associated with the β-phase. It has value in the void space only. It can be a scalar, like for instance the pressure field, p_β or a vector, as the velocity field, $\mathbf{v}_\beta$. We introduce the average over the control volume, V, known as superficial average and defined as

$$\langle \phi_\beta \rangle = \frac{1}{V} \int_{V_\beta} \phi_\beta dV, \tag{1.3}$$

and the intrinsic phase average as

$$\langle \phi_\beta \rangle^\beta = \frac{1}{V_\beta} \int_{V_\beta} \phi_\beta dV. \tag{1.4}$$

For instance, the Darcy velocity, $\langle \mathbf{v}_\beta \rangle$, is defined with this first average while the macroscale pressure, $\langle p_\beta \rangle^\beta$

is the average value in the void space only. Both operators are merely related by the trivial relation,

$$\langle \phi_\beta \rangle = \varepsilon \langle \phi_\beta \rangle^\beta, \tag{1.5}$$

where $\varepsilon \equiv \frac{V_\beta}{V}$ is the porosity, which is the ratio of the volume occupied by the void (the pore volume) over the control volume (bulk volume). It measures the void spaces in a material that can be occupied by fluids. In most cases the porous medium is assumed to be motionless and non-deformable, which means that ε does not vary over time and only depends on the solid geometry.

Now that the average operators are defined, and assuming that the REV exists, we can give a mathematical definition of porosity. It is based on the distribution of the void and solid that coexist in the control volume. Lets define the mask function, χ_β, in the entire domain V as,

$$\chi_\beta = \begin{cases} 1 & \text{in } V_\beta \\ 0 & \text{elsewhere} \end{cases}. \tag{1.6}$$

This function labels the void with 1 and the solid phase with 0. It can results from the process of segmentation of a raw image which allows to differentiating the solid from the void (see Figure 1.2). The integration of this function over a control volume, V, yields in the volume, $V_\beta = \int_V \chi_\beta dV$, occupied by the void within V. The volume fraction of void or *porosity*, ε, is merely described as the superficial average of the indicator function[2],

$$\varepsilon = \langle \chi_\beta \rangle. \tag{1.7}$$

Actually, we can think of $\langle . \rangle$ as an average operator that can be applied to Stokes equations in order to derive the governing equations for the average values. This upscaling process, however, is not straightforward. In particular, the average of a derivative is the derivative of the average plus the contribution of the boundary values at the solid walls. Mathematically it is performed with the following homogeneization theorem,

$$\langle \nabla \phi_\beta \rangle = \nabla \langle \phi_\beta \rangle + \frac{1}{V} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \phi_\beta dA, \tag{1.8}$$

where $\mathbf{n}_{\beta\sigma}$ denotes the normal vector at the solid walls. The surface integral term in Eq (1.8) translates the boundary value of the direct model to a volumetric source term in the Darcy-scale representation. This term is essential and has to be taken into account. More details on this volume averaging method is given in Chapter 7.

It took a long time to relate Darcy's law to the equations of fluid mechanics. This has been performed over the past decades by integrating the Stokes equations over a representative volume element of the porous medium (see for instance the work of Whitaker [6] who derived Darcy's law from the volume averaging of Stokes). These major breakthroughs in theoretical fluid mechanics brought new perspectives to Darcy's law. Actually, when following the theoretical upscaling from Stokes momentum equation, the resulting equation is not Darcy, but a more comprehensive conservation law first proposed by Brinkman [1] and referred to as Darcy-Brinkman or Darcy-Brinkman-Stokes equation. It reads,

$$0 = -\nabla \langle p_\beta \rangle^\beta + \rho_\beta \mathbf{g} + \mu_\beta^* \nabla^2 \langle \mathbf{v}_\beta \rangle - \mu_\beta \mathsf{K}^{-1} . \langle \mathbf{v}_\beta \rangle. \tag{1.9}$$

[2]Note that by definition the intrinsic volume average of χ_β is always $\langle \chi_\beta \rangle^\beta = 1$.

One can recognize the Stokes equation for the average velocity and pressure with an additional source term, $\mu_\beta \mathsf{K}^{-1}.\langle \mathbf{v}_\beta \rangle$. From a fluid mechanics point of view, this term is a drag force due to the viscous friction of the fluid with the walls of the solid structure. It comes from the surface integral in Eq (1.8). It is the same force that gives a cyclist hard time when he bikes against the wind (see Figure 1.4). Therefore, the permeability is seen as a drag force coefficient. Compared to a bicycle, the solid surface in a porous medium can be very large, and so are the friction forces. The larger is the solid surface, the more important is the drag force. In practice, it turns out that most of the time, the viscous term, $\mu_\beta^* \nabla^2 \langle \mathbf{v}_\beta \rangle$, is negligible compared with $\mu_\beta \mathsf{K}^{-1}.\langle \mathbf{v}_\beta \rangle$, which leads then to Darcy's law. We will see in Chapter 4 some situations where it is relevant to use Darcy-Brinkman equation.

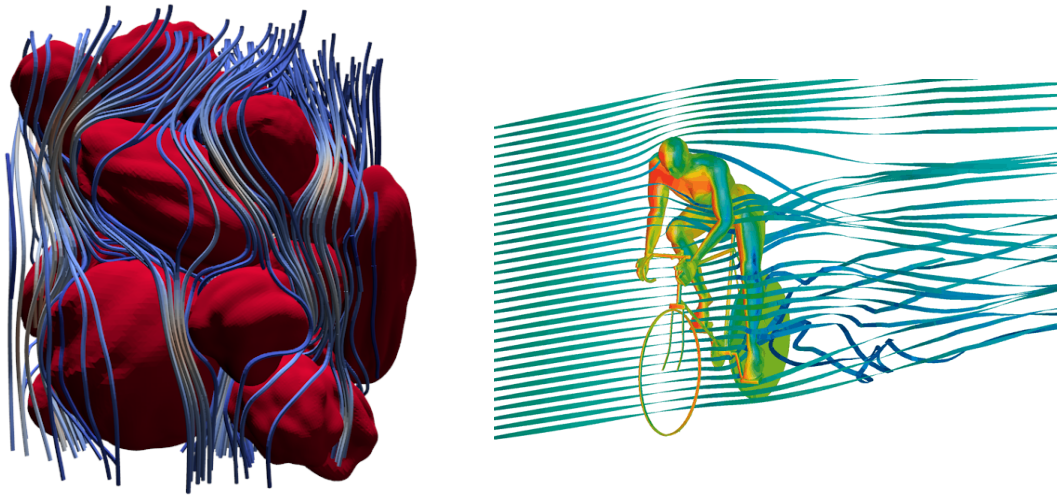

Figure 1.4: Streamlines through a porous medium and a cyclist. In both cases, the friction of the fluid against the solid surfaces yields in drag forces.

Perhaps even more important than just the mathematical demonstration, the homogeneization works that have lead to Darcy's equation emphasized that this law, which was initially formulated as a phenomenogical law for natural porous media with small pore throats can be extended to other porous media. All the media depicted in Figure 1.5 can be considered as porous media, with characteristic pore throat sizes ranging from few microns to few meters. For instance, the structured packings that equip air distillation columns for air separation or CO_2 capture are made of the same elementary pattern with pores of about a centimeter that are repeated millions of time within the column. The highest columns can reach up to 60 m high. It is clear that a pore-scale representation is not sustainable to simulate the flow in the entire column and that an averaged approach based on Darcy's law is more relevant [4]. For the same reason when investigating forest fires at large scale, it is not conceivable to represent all the branches and leaves, but instead, the canopy is considered as a porous medium where Darcy's law is valid.

1.3 Exercises

In some particular configurations, the Stokes equations have simple analytical solutions as, for instance, in the case of viscous flow between two parallel plates or flow through a capillary tube.

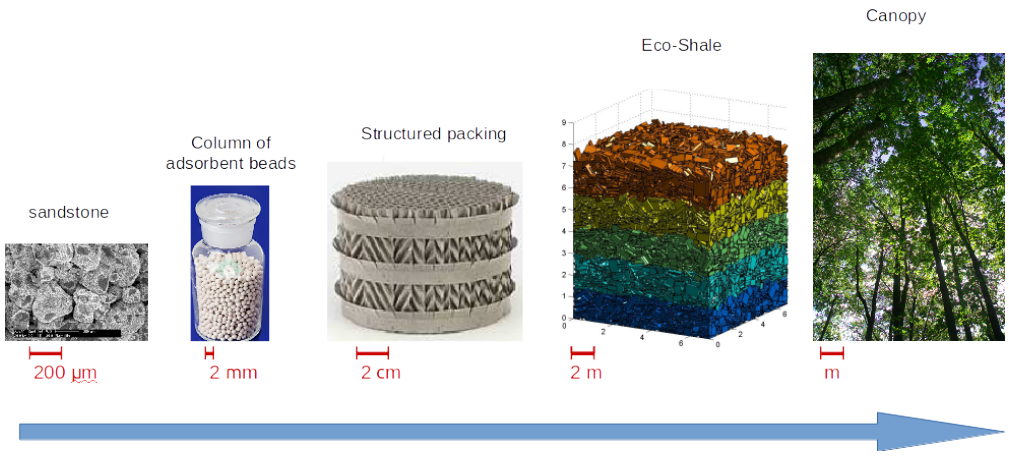

Figure 1.5: Example of porous media for a wide range of length scales.

1.3.1 Viscous flow between parallel plates

In the case of viscous flow between two parallel plates (see Figure 1.6), we can suppose that there is a uniform effective pressure gradient in the x-direction, so that $\nabla p = \frac{\Delta P}{L}\mathbf{e}_x$. Moreover, considering that the

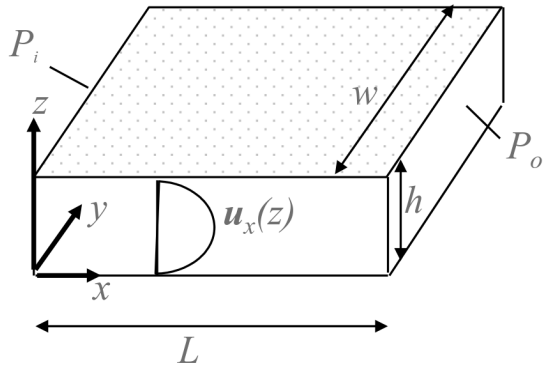

Figure 1.6: Schematic of flow between two parallel plates

dimensions of the plates are much larger than the thickness h, then we can consider that the flow is only carried by the x component and only depends on z, *i.e.*,$\mathbf{v}_\beta = v_x(z)\mathbf{e}_x$. Hence, the momentum equation reads

$$\mu_\beta \frac{\partial^2 v_x}{\partial z^2} = \frac{\Delta P}{L}, \tag{1.10}$$

with the non-slip boundary condition at the walls,

$$v_x\left(z = \pm\frac{h}{2}\right) \quad = \quad 0. \tag{1.11}$$

The integration of this boundary value problem leads to the following parabolic profile,

$$v_x(z) = \frac{1}{\mu_\beta}\frac{h^2}{8}\frac{\Delta P}{L}\left[1-\left(\frac{2z}{h}\right)^2\right]. \tag{1.12}$$

This is the 2D solution of a Poiseuille flow. We can notice that the maximum velocity, $v_{z,max} = \frac{h^2}{8\mu_\beta}\frac{\Delta P}{L}$, is reached in the middle of the channel. Though very simple, this parabolic equation has lot of applications in porous media modeling. If it is averaged over the thickness, we obtained the Hele-Shaw relation [3],

$$\langle v_x \rangle = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} v_x(z)\,dz = \frac{h^2}{12\mu_\beta}\frac{\Delta P}{L}. \tag{1.13}$$

The Hele-Shaw cells consist of two parallel plates fixed at a small distance sandwiching a viscous fluid. They are widely used in fluid mechanics experiments to visualize the flow patterns. In a sense, the micromodels are Hele-Shaw cells with obstacles to mimic a porous medium. This Hele-Shaw equation that relates the average velocity to the pressure gradient reminds us of Darcy's law with a permeability $K = \frac{h^2}{12}$ and a porosity $\varepsilon = 1$. Actually, this simple integration was historically the first step towards the derivation of Darcy's law from Stokes momentum equation (see [5] for an interesting history review from Hele-Shaw's first experiments to Darcy's law formulation).

Equation 1.12 also has applications for flow in fractured media. Indeed, at leading order, the fluid flow through fractures is conceptualized by using the assumption of laminar flow between parallel plates. The total flow between parallel, $q = \langle v_x \rangle hw$ is equal to,

$$q = \frac{h^3 w}{12\mu_\beta}\frac{\Delta P}{L}.$$

We recognize, here, the cubic law where h represents the fracture aperture.

1.3.2 Viscous flow through a capillary

Another important analytical solution of the Navier-Stokes equations is the Poiseuille flow in a capillary, see Figure 1.7. This solution, first proposed by the physicist and physiologist Poiseuille for the study of blood flows in capillary, has a lot of impact on porous media research since it is at the origin of the well-known Kozeny-Carman correlation to estimate the permeability according to the porosity.

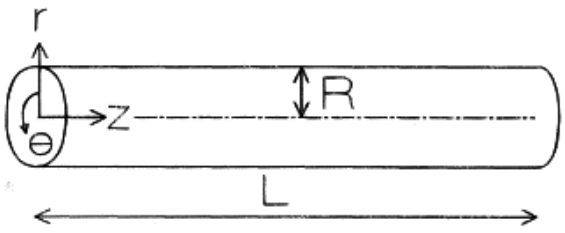

Figure 1.7: Schematic of flow through a capillary

Considering that the tube is long enough compared to its radius and due to axial symmetry, we can assume that $\nabla p = \frac{\Delta P}{L}\mathbf{e}_z$ and that the velocity is only carried by the z component that only depends on the

radial positon,$\mathbf{v}_\beta = v_z(r)\mathbf{e}_z$. In cylindrical coordinates, the Stokes momentum equation becomes,

$$\mu_\beta \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{\Delta P}{L}, \tag{1.14}$$

with the non-slip boundary conditions at the walls,

$$v_z(r = R) = 0.$$

The integration of this boundary values problem leads to the following parabolic profile,

$$v_z(r) = \frac{R^2}{4\mu_\beta}\left(1 - \left(\frac{r}{R}\right)^2\right)\frac{\Delta P}{L}.$$

It is the 3D solution of a Poiseuille flow. We can notice that the maximum velocity, $v_{z,max} = \frac{R^2}{4\mu_\beta}\frac{\Delta P}{L}$, is reached in the middle of the capillary. The intrinsic average of this equation yields in,

$$\langle v_z\rangle^\beta = \frac{2}{R^2}\int_0^R v_z\left(r\right) r dr = \frac{R^2}{8\mu_\beta}\frac{\Delta P}{L}.$$

The total flow through the capillary tube, $q = \langle v_z\rangle^\beta \pi R^2$ is equal to,

$$q = \frac{\pi R^4}{8\mu_\beta}\frac{\Delta P}{L}.$$

We recognize, here, the Hagen-Poiseuille equation.

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